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New Variational Perturbation Theory Based on $q-$Deformed Oscillator

Hyeong-Chan Kim,[1,3] Jae Hyung Yee,[1] and Sang Pyo Kim[2]

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A new variational perturbation theory is developed based on the $q-$deformed oscillator. It is shown that the new variational perturbation method provides 200 and 10 times better accuracy for the ground state energy of anharmonic oscillator than the Gaussian and the post Gaussian approximation, respectively, for weak coupling.

KEY WORDS: $q-$deformation; anharmonic oscillator; nonperturbative approximation.

There have been many proposals to establish methods of extracting nonperturbative information from quantum mechanical and quantum field theoretic systems, such as $1/N$ expansion and methods based on Dyson–Schwinger equation (Zinn-Justin, 1996). One of the successful ways to obtain nonperturbative information is the variational perturbation theory (Jackiw and Kerman, 1979; Stevenson, 1984). Many variants of the variational perturbation theory based on the Gaussian approximation have been used to study various aspects of physical systems (Amore *et al*., 2004; Bak, *et al*., 2000; Cea and Tedesco, 1997; Kleinert, 1993, 1995; Kim, *et al*., 2004, 2005; Lee and Yee, 1997; Lee *et al*., 1998; Okopinska, 1987,1996; You *et al*., 1998). Although the variational perturbation theory provides one with systematic correction terms to the variationally determined approximation, it has a limitation that only the Gaussian wave function(al) may be used as a variational trial wave function(al) for the most physical systems. In this paper we attempt to establish a new variational perturbation theory based on the $q-$deformed oscillator (Bonatsos and Daskaloyannis, 1997; Macfarlane, 1989), which provides a better approximation than that based on the Gaussian approximation.

Quantum anharmonic oscillator has been frequently used in developing various approximation methods in quantum mechanics and quantum field theory (Bak

[1] Department of Physics, Yonsei University, Seoul 120-749, Korea.
[2] Department of Physics, Kunsan National University, Kunsan 573-701, Korea.
[3] To whom correspondence should be addressed at Department of Physics, Yonsei University, Seoul 120-749, Korea; e-mail: hckim@phya.yonsei.ac.kr.

et al., 2000; Cea and Tedesco, 1997; Dineykhan *et al*., 1995; Kleinert, 1993, 1995; Lee and Yee, 1997; Lee *et al*., 1997; You *et al*., 1998; Zinn-Justin, 1996). To develop a new variational perturbation method we will consider the anharmonic oscillator given by the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2}{2}\hat{x}^2 + \frac{\lambda}{4}\hat{x}^4. \tag{1}$$

We first illustrate some essential procedures of the variational perturbation theory based on the Gaussian approximation. We start by writing the Hamiltonian (1) as the sum of the Gaussian Hamiltonian, H_G, and the perturbation term, V_I, as

$$\hat{H} = \hat{H}_G + V_I, \tag{2}$$

where

$$\hat{H}_G \equiv \frac{1}{2}\hat{p}^2 + \frac{\Omega_G^2}{2}\hat{x}^2, \quad V_I = \frac{\omega^2 - \Omega_G^2}{2}\hat{x}^2 + \frac{\lambda}{4}\hat{x}^4. \tag{3}$$

Since the Gaussian Hamiltonian is that of a simple harmonic oscillator, it can be written in terms of the creation and the annihilation operator, $\hat{A}_G = \sqrt{\frac{\Omega_G}{2\bar{h}}}\hat{x} + \frac{i}{\sqrt{2\bar{h}\Omega_G}}\hat{p}$, in the quadratic form:

$$\hat{H}_G = \frac{\bar{h}\Omega_G}{2}\left(\hat{A}_G^\dagger \hat{A}_G + \hat{A}_G \hat{A}_G^\dagger\right). \tag{4}$$

Moreover, the Lie algebra, $[\hat{A}_G, \hat{A}_G^\dagger] = 1$, allows one to define a Fock space consisted of the ground state $|0\rangle_G$ defined by,

$$\hat{A}_G|0\rangle_G = 0, \tag{5}$$

and the excited states, $|n\rangle_G$, generated by successively acting $A_G^\dagger$ on $|0\rangle_G$.

The next step is the variational procedure with respect to Ω_G so that the energy of the Hamiltonian (2) with respect to the Gaussian ground state is minimized. Some of the nonperturbative effects of the Hamiltonian (1) are amalgamated to $\hat{H}_G$ by this process, which makes the correction term, V_I, to be an order of ξ_G (defined below) smaller than the Gaussian Hamiltonian. The V_I term may be written as in Bak *et al*. (1999, 2000) as

$$V_I = \frac{\xi_G \bar{h}\Omega_G}{2}\left[-1 + \frac{1}{3}\sum_{k=0}^{4}\binom{4}{k}\hat{A}_G^{\dagger k}\hat{A}_G^{4-k}\right], \tag{6}$$

where $\xi_G\left(=\frac{3\lambda\bar{h}}{2\Omega_G^3}\right) < 1$ for any value of λ due to the variational gap equation, $1 - \xi_G = \frac{\omega^2}{\Omega_G^2}$. In this sense, the Gaussian Hamiltonian $\hat{H}_G$ describes a truncated form of $\hat{H}$ up to $O(\xi_G^{\,0})$.

The final step in the variational perturbation theory is to apply the conventional perturbation method to the Hamiltonian (2) by using the complete set of states, $\{|n\rangle_G\}$. In an operator method, the so called Liouville-von Neumann approach, this is achieved by defining the 1st order annihilation operator, $\hat{A}_{(1)} = \hat{A}_G + \xi_G \hat{B}$, so that the Hamiltonian $\hat{H}$ can be factorized as $\hat{H} = \frac{\bar{h}\Omega}{2}(\hat{A}^\dagger_{(1)}\hat{A}_{(1)} + \hat{A}_{(1)}\hat{A}^\dagger_{(1)})$, up to the first order in ξ_G (Bak *et al*., 1999, 2000). It was shown that these operators satisfy the (real) $q-$deformed algebra (Bonatsos and Daskaloyannis, 1999) with deformation parameter, $q^2 = 1 + \xi_G/2$. This algebra leads to another Fock space determined by the 1st order creation operator, $\hat{A}^\dagger_{(1)}$, which defines the 1st order states $|n\rangle_{(1)}$.

The fact that the first order perturbative correction to the Gaussian approximation is described by the $q-$deformed oscillator indicates that one may establish a better variational perturbation theory by using the $q-$deformed oscillator as the basis for the variational method. To develop the variational perturbation method based on the $q-$deformed oscillator, we start by separating the Hamiltonian (1) into the $q-$deformed part, $\hat{H}_q$, and the perturbation term, V_I':

$$\hat{H} = \hat{H}_q + V_I', \tag{7}$$

where $V_I' = \hat{H} - \hat{H}_q$, and

$$\hat{H}_q \equiv \frac{\bar{h}\bar{\Omega}}{2}(\hat{a}_q\hat{a}_q^\dagger + \hat{a}_q^\dagger\hat{a}_q) \tag{8}$$

is the Hamiltonian of the $q-$deformed oscillator (Bonatsos and Daskaloyannis, 1999), with $\hat{a}_q$ and $\hat{a}_q^\dagger$ satisfying the algebra,

$$[\hat{a}_q, \hat{a}_q^\dagger] = 1 + \epsilon\hat{a}_q^\dagger\hat{a}_q, \tag{9}$$

where we use this specific form of $q-$algebra, since this is satisfied with the perturbative anharmonic oscillator (Bak *et al*., 1999, 2000). The $q-$ground state is defined by

$$\hat{a}_q|0\rangle_q = 0, \tag{10}$$

and the $q-$excited states can be generated by successively acting $\hat{a}_q^\dagger$ to the ground state (Amore *et al*., 2004).

For calculational simplicity, we introduce dimensionless operator $\hat{H}^n_m$ of order $O(\bar{h}^0)$:

$$\hat{H}^n_m = \left(\frac{\bar{h}}{2\Omega_q}\right)^{-\frac{m}{2}} \left(\frac{\Omega_q\bar{h}}{2}\right)^{-\frac{n}{2}} \frac{\hat{x}^m\hat{p}^n + \hat{p}^n\hat{x}^m}{2}. \tag{11}$$

Similar operators are defined in Bender and Dunne (1989). The explicit expression for $\hat{a}_q$, which satisfies the algebra (9) and make the Hamiltonian (8) to be (A.4) is

obtained to $O(\epsilon^2)$ in Appendix A,

$$\hat{a}_q = u^* \left\{ i\hat{H}_0^1 + \left(1 + \frac{\epsilon^2}{4}\right)\hat{H}_1^0 + \frac{\epsilon(1-\epsilon/2)}{4}\left[\hat{H}_1^2 - i\hat{H}_2^1 + \frac{2}{3}\hat{H}_3^0\right] + \frac{\epsilon^2}{32}\left[-i\hat{H}_2^3 - \hat{H}_3^2 - i\frac{4}{3}\hat{H}_4^1 + \frac{8}{5}\hat{H}_5^0\right]\right\} + \cdots, \tag{12}$$

where $u^* = \frac{1}{2}\left(1 - \frac{1}{4}\epsilon - \frac{9}{32}\epsilon^2\right)$. After inverting the expression for $\hat{a}_q$ to get $\hat{H}_1^0$ as a function of $\hat{a}_q$ and $\hat{a}_q^\dagger$, and taking the expectation value with respect to the $q-$ground state (10), we obtain

$$\left\langle \hat{H}_2^0 \right\rangle_q = 1 - \epsilon + \frac{13}{24}\epsilon^2, \quad \left\langle \hat{H}_4^0 \right\rangle_q = 3 - 7\epsilon + \frac{33}{4}\epsilon^2, \quad \left\langle \hat{H}_6^0 \right\rangle_q = 15. \tag{13}$$

Note that we have introduced two variational parameters $\bar{\Omega}$ and ϵ. To establish the variational perturbation theory based on the $q-$Hamiltonian, we need to express the $q-$Hamiltonian (8) in terms of the phase space variables, $\hat{x}$ and $\hat{p}$. By demanding that the $q-$Hamiltonian is frictionless (no $\hat{p}\hat{x} + \hat{x}\hat{p}$ term), the $q-$Hamiltonian may be written as, up to $O(\epsilon^2)$,

$$\hat{H}_q = \frac{\bar{h}\Omega_q}{4}\left[\hat{H}_2^0 + \hat{H}_0^2 + \frac{\epsilon}{3}\left(1 - \frac{\epsilon}{2}\right)\hat{H}_4^0 - \frac{g_3\epsilon^2}{4}\hat{H}_6^0 + \cdots\right]. \tag{14}$$

The parameters in (8) and (14) are determined in the Appendix A and $g_3 = -\frac{23}{45}$.

Note that, unlike the case of the variational perturbation theory based on the Gaussian approximation, the $q-$Hamiltonian $\hat{H}_q$ is not expressed in a closed form in the phase space, but is written as a series in ϵ. Since we are to compute up to the 1st order correction to the variational result, it is enough to write $\hat{H}_q$ up to $O(\epsilon^2)$ as in Eq. (14).

The Hamiltonian $\hat{H}$ of Eq. (1) is then written as, $\hat{H} = \hat{H}_q + V_I'$, up to $O(\epsilon^2)$, where

$$V_I' = \frac{\bar{h}\Omega_q}{4}\left[\left(\frac{\omega^2}{\Omega_q^2} - 1\right)\hat{H}_2^0 + \frac{\xi - 2\epsilon + \epsilon^2}{6}\hat{H}_4^0 + \frac{g_3\epsilon^2}{4}\hat{H}_6^0 + \cdots\right], \tag{15}$$

with $\xi = \frac{3\lambda\bar{h}}{2\Omega_q^3}$. Taking expectation value of (7) with respect to the $q-$ground state (10), the energy expectation value of the anharmonic oscillator becomes

$${}_q\langle 0|\hat{H}|0\rangle_q = \frac{\bar{h}\Omega_q}{4}\left[1 + \epsilon + \frac{5}{8}\epsilon^2 + \left(1 - \epsilon + \frac{13}{24}\epsilon^2\right)\left(\frac{\xi}{\xi_0}\right)^{2/3} + \left(1 - \frac{7}{3}\epsilon + \frac{11}{4}\epsilon^2\right)\frac{\xi}{2}\right] + O(\epsilon^3). \tag{16}$$

The variational minimization of the energy expectation value (16), with respect to Ω_q and ϵ, leads to the gap equations,

$$\left(1-\epsilon+\frac{13}{24}\epsilon^2\right)\left(\frac{\xi}{\xi_0}\right)^{2/3}=1+\epsilon+\frac{5}{8}\epsilon^2-\left(1-\frac{7}{3}\epsilon+\frac{11}{4}\epsilon^2\right)\xi,$$

$$\frac{5}{4}\left[1+\frac{13}{15}\left(\frac{\xi}{\xi_0}\right)^{2/3}+\frac{11}{5}\xi\right]\epsilon=\left(\frac{\xi}{\xi_0}\right)^{2/3}+\frac{7}{6}\xi-1, \tag{17}$$

where $\xi_0=\dfrac{3\lambda\bar{h}}{2\omega^3}$. The gap equations relate ξ to ϵ as,

$$\xi=2\epsilon\left(1-\frac{3\epsilon}{4}+\frac{65\epsilon^2}{16}\right)f, \tag{18}$$

$$f=\left(1-3\epsilon-\frac{91\epsilon^2}{8}+\frac{143\epsilon^3}{16}\right)^{-1}. \tag{19}$$

By using Eq. (17), the ground state energy of the anharmonic oscillator becomes

$$E_0=\frac{\bar{h}\omega}{2}\left(\frac{\xi_0}{\xi}\right)^{1/3}\left[1+\epsilon+\frac{5}{8}\epsilon^2-\left(1-\frac{7}{3}\epsilon+\frac{11}{4}\epsilon^2\right)\frac{\xi(\epsilon)}{4}\right]+O(\epsilon^3),$$

where $\xi(\xi_0)$ and $\epsilon(\xi_0)$ are determined by the gap Eq. (17). We present the values of ϵ, ξ, and E_0 for several values of the coupling, ξ_0, in Table I. As can be seen in this Table, the maximal error of the present approximation is less than 0.8%. Moreover, the accuracy is 200 times and 10 times better than the Gaussian and the post (including the 2nd order perturbative corrections) Gaussian approximations (Lee and Yee, 1997), respectively, for weak coupling. The maximum value of the dimensionless expansion parameter, $\epsilon\simeq 0.1717$, is attained as $\xi_0\to\infty$, which is distinguishably small compared to the Gaussian expansion parameter, $\xi_G=1$, at the same limit. This clearly shows that the present method provides better nonperturbative information than the Gaussian approximation and 2nd order variational perturbation results based on the Gaussian approximation.

To complete the 1st order perturbative corrections to the variational result, we need to construct the 1st order creation and annihilation operators, which are correct to $O(\epsilon^2)$. To do this, we need to express the Hamiltonian (7) as a function of $q-$operators, $\hat{a}_q$ and $\hat{a}_q^\dagger$. Equation (18) enables one to write the coefficients in the potential V_I', Eq. (15), as

$$\frac{\omega^2}{\Omega_q^2}-1=\frac{\epsilon^2}{4}g_1(\epsilon),\quad \frac{\xi-2\epsilon+\epsilon^2}{6}=\frac{\epsilon^2}{4}g_2(\epsilon), \tag{20}$$

$$g_1(\epsilon)=\left(1-\frac{329}{2}\epsilon\right)f,\quad g_2(\epsilon)=3\left(1+\frac{247}{36}\epsilon-\frac{143}{36}\epsilon^2\right)f+\frac{2}{3}.$$

Table I. Ground State Energy of Anharmonic Oscillator

	Gaussian		Post gaussian(2nd)		Present method			
ξ_0	$E_0/(\bar{h}\omega)$	error(%)	$E_0/(\bar{h}\omega)$	error(%)	$E_0/(\bar{h}\omega)$	error(%)	ξ	ϵ
	1/2	0	1/2	0	1/2	0	0	0
0.06	0.507	0.006	0.507	−0.0003	0.507	0.00003	0.0599	0.028
0.6	0.560	0.21	0.559	−0.037	0.559	0.0083	0.481	0.121
6	0.813	1.09	0.801	−0.37	0.806	0.307	1.264	0.162
60	1.531	1.75	1.49	−0.68	1.514	0.628	1.627	0.170
600	3.19	1.95	3.11	−0.79	3.15	0.731	1.722	0.171
∞	$0.375\ \xi_0^{1/3}$	2.01	$0.365\ \xi_0^{1/3}$	−0.82	$0.370\xi_0^{1/3}$	0.766	1.75	0.172

Note that $g_i(\epsilon)$'s are of $O(1)$, and thus this shows the fact that V_I' is order of ϵ smaller than $\hat{H}_q$. The Hamiltonian is then written as,

$$\hat{H} = \frac{\bar{h}\bar{\Omega}}{2}[\hat{a}_q\hat{a}_q^\dagger + \hat{a}_q^\dagger\hat{a}_q] + \frac{\bar{h}\Omega_q\epsilon^2}{16}\sum_{i=1}^{3} g_i(\epsilon)\hat{H}_{2i}^0 + \cdots \tag{21}$$

$$= \frac{\bar{h}\bar{\Omega}}{2}[\hat{a}_q\hat{a}_q^\dagger + \hat{a}_q^\dagger\hat{a}_q] + \frac{\bar{h}\Omega_q\epsilon^2}{16}\{g_1(\epsilon) + 3g_2(\epsilon) + 15g_3 + [g_1(\epsilon)$$

$$+ 6g_2(\epsilon) + 45g_3]\sum_{r=0}^{2}\begin{pmatrix} 2 \\ r \end{pmatrix}(\hat{a}_q^\dagger)^{2-r}(\hat{a}_q)^r$$

$$+ (g_2 + 15g_3)\sum_{r=0}^{4}\begin{pmatrix} 4 \\ r \end{pmatrix}(\hat{a}_q^\dagger)^{4-r}(\hat{a}_q)^r$$

$$+ g_3\sum_{r=0}^{6}\begin{pmatrix} 6 \\ r \end{pmatrix}(\hat{a}_q^\dagger)^{6-r}(\hat{a}_q)^r\Bigg\} + O(\epsilon^3).$$

We now want to write this Hamiltonian as a generalized deformed oscillator:

$$\hat{H} = \frac{\bar{h}\Omega}{2}\left(\hat{a}_{(1)}\hat{a}_{(1)}^\dagger + \hat{a}_{(1)}^\dagger\hat{a}_{(1)}\right) + O(\epsilon^3), \tag{22}$$

$$\left[\hat{a}_{(1)}, \hat{a}_{(1)}^\dagger\right] = 1 + \epsilon\alpha_1\hat{a}_{(1)}^\dagger\hat{a}_{(1)} + \epsilon^2\alpha_2\left(\hat{a}_{(1)}^\dagger\hat{a}_{(1)}\right)^2,$$

where the algebra defines the deformation function, $F(y) = 1 + (1 + \epsilon\alpha_1)y + \alpha_2(\epsilon y)$.[4] Since V_I' is $O(\epsilon^2)$, correction to the annihilation operator would be of

[4] The function $F(y)$ is related to $\Phi(y)$ of Bonatsos and Daskaloyannis (1999) by $F(y) = \Phi(\Phi^{-1}(y) + 1)$.

order ϵ^2, and thus, $\hat{a}_{(1)}$ can be written as

$$\hat{a}_{(1)} = \hat{a}_q + \epsilon^2 \left[\sum_{n=0}^{1} u_n (\hat{a}_q^\dagger)^{1-n} (\hat{a}_q)^n + \sum_{n=0}^{3} v_n (\hat{a}_q^\dagger)^{3-n} (\hat{a}_q)^n + \sum_{n=0}^{5} w_n (\hat{a}_q^\dagger)^{5-n} (\hat{a}_q)^n \right] + \cdots . \tag{23}$$

For this operator to satisfy the algebra (22), the following relations should hold for the coefficients in Eq. (23):

$$\begin{aligned} & w_0 = \frac{2}{15} w_1 = \frac{w_2}{24} = \frac{w_3 + w_3^*}{20} = \frac{g_3}{16} \frac{\Omega_q}{\Omega}, \quad w_4 = -\frac{w_2}{2}, \quad w_5 = -\frac{w_1}{5}, \\ & v_0 = \frac{v_1}{6} = (g_2 - 10 g_3) \frac{\Omega_q}{16\Omega}, \quad v_2 + v_2^* = (g_2 + 10 g_3) \frac{3\Omega_q}{8\Omega}, \quad v_3 = -\frac{v_1}{3}, \\ & u_1 + u_1^* = 0, \quad u_0 = \frac{\Omega_q}{16\Omega} (g_1 + 6 g_2 + 45 g_3), \end{aligned} \tag{24}$$

which give, to $O(\epsilon^2)$,

$$\begin{aligned} \Omega &= \left[1 + \frac{\epsilon^2}{8} (g_1 + 3 g_2 + 15 g_3) \right] \bar{\Omega}, \\ \alpha_1 &= 1 + \frac{3\epsilon}{4} (g_2 + 5 g_3), \quad \alpha_2 = \frac{15}{4} g_3. \end{aligned} \tag{25}$$

To obtain the 1st order correction to $O(\epsilon^2)$, we may set $\Omega_q / \Omega \to 1$ in Eq. (24), since all the coefficients of Ω_q / Ω have factors of order $O(\epsilon^2)$. The 1st order ground state is then defined by

$$\hat{a}_{(1)} |0\rangle_{(1)} = 0, \tag{26}$$

and its energy is the same as that in Eq. (20), since the variational approximation leading to (20) includes the contribution from the potential V_I', as in the case of the Gaussian variational perturbation. The nth excited states are given by

$$|n\rangle_{(1)} = \frac{(\hat{a}_{(1)}^\dagger)^n}{\sqrt{[n]!}} |0\rangle_{(1)}, \tag{27}$$

where $[n]! = [n][n-1] \cdots [1]$, and $[n]$ is defined by the recurrence relation $[n+1] = F([n])$, with $[0] = 0$. The energy of the nth eigenstate is given by

$$E_n = \frac{\bar{h}\Omega}{2} \left([n] + [n+1] \right). \tag{28}$$

In contrast to the ground state energy that shows a slight improvement for large ξ_0, the energies of the excited states receive considerable improvements compared to

those in the Gaussian and the post Gaussian approximation, since the expansion parameter ϵ is smaller than that of the Gaussian approximation.

The higher order corrections can be obtained similarly as in the above procedure. To obtain the 2nd order correction, for example, we need to write the Hamiltonian (21) up to $O(\epsilon^3)$, and to express it as a function of $\hat{a}_{(1)}$ and $\hat{a}^\dagger_{(1)}$. We then write the Hamiltonian in a factorized form, $H_{(2)} = \frac{\bar{h}\Omega}{2}(\hat{a}_{(2)}\hat{a}^\dagger_{(2)} + \hat{a}^\dagger_{(2)}\hat{a}_{(2)}) + O(\epsilon^4)$, while $\hat{a}_{(2)}$ satisfies a generalized deformed algebra.

The fact that the first order perturbative result can be expressed as a generalized deformed oscillator, Eq. (22), also provides us with a possibility of establishing a new variational perturbation theory based on the generalized deformed oscillator (22).

One of the main advantage, for using the algebraic approach is that, it is easy to define the thermal state and coherent state. For example, the coherent state can be written as the state $|\alpha\rangle$,

$$\hat{a}(t)|\alpha\rangle = |\alpha\rangle\alpha. \tag{29}$$

The construction of the state $|\alpha\rangle$ from the number states can be achieved from the usual method of the $q-$oscillator (Cho *et al*., 1994).

It would be interesting to apply the present method to general quantum mechanical systems. For example, the realization of $\mathrm{su}_q(2)$ by $q-$boson can be used as a basis of a perturbation theory for systems with spherical symmetric potential. More interesting would be to generalize the present method to quantum field theory by expanding the quantum fields in Fourier modes where each mode acts as a generalized deformed oscillator.

APPENDIX A: CALCULATION OF THE ANNIHILATION OPERATOR

To have an explicit expression for the annihilation operator $\hat{a}_q$, we need the $q-$annihilation operator as a function of $\hat{x}$ and $\hat{p}$,

$$\hat{a}_q = \sum_{l=0}\sum_{n=0}^{2l+1} \epsilon^l u^*_{n,l} \hat{H}_n^{2l+1-n}, \tag{A.1}$$

which satisfies Eqs. (8) and (9). In obtaining this expression, the operator product expansion formula,

$$\hat{H}_m^n \hat{H}_{m'}^{n'} = \sum_{k=0} i^k h_k(m, n; m', n') \hat{H}_{m+m'-k}^{n+n'-k} \tag{A.2}$$

is useful. The expansion coefficients, h_k are given as

$$h_0 = 1, \quad h_1 = mn' - m'n,$$

$$
\begin{aligned}
h_2 &= -\left[m(m-1)+m'(m'-1)\right]nn' - mm'\left[n(n-1)\right. \\
&\quad \left. + n'(n'-1)\right] - 3mm'nn', \\
h_3(m,n;m',n') &= 2\left(f_m^{n'} - f_{m'}^{n}\right) + \left(f_{m+m'}^{n} - f_m^{n+n'}\right) - 3m'm(m-1) \\
&\quad \times \begin{pmatrix} n+n' \\ 3 \end{pmatrix} + 3n'n(n-1)\begin{pmatrix} m+m' \\ 3 \end{pmatrix} \\
&\quad -3\left[m(m-1)n'(n'-1) - m'(m'-1)n(n-1)\right] \\
&\quad \times (m+m'-2)(n+n'-2),
\end{aligned} \tag{A.3}
$$

where $f_m^n = 6\begin{pmatrix} m \\ 3 \end{pmatrix}\begin{pmatrix} n \\ 3 \end{pmatrix}$.

We assume that the Hamiltonian for the q-oscillator is written as the sum of the kinetic energy and potential term dependent only on x,

$$
\hat{H}_q = \frac{\hat{p}^2}{2} + \frac{\Omega_q^2}{2}x^2 + \frac{c}{4}x^4 + \frac{d}{6}x^6 + \cdots. \tag{A.4}
$$

If we demand the operator $\hat{a}_q$ to satisfy the algebra (9) and the Hamiltonian (8) to be (A.4) we get $\hat{a}_q$ in Eq. (12) and the $q-$Hamiltonian

$$
\hat{H}_q = \frac{\bar{h}\Omega_q}{4}\left[\hat{H}_2^0 + \hat{H}_0^2 + \frac{\epsilon}{3}\left(1-\frac{\epsilon}{2}\right)\hat{H}_4^0 - \frac{g_3\epsilon^2}{4}\hat{H}_6^0 + \cdots\right]. \tag{A.5}
$$

The parameters in (8) and (14) are determined as

$$
\bar{\Omega} = \left(1 + \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2\right)\Omega_q, \quad c = \frac{4\epsilon\Omega_q^3}{3\bar{h}}\left(1-\frac{\epsilon}{2}\right), \tag{A.6}
$$

$$
d = -\frac{3\epsilon^2 g_3 \Omega_q^4}{\bar{h}^2}, \; g_3 = -\frac{23}{45}.
$$

The present result can be easily checked using mathematica or maple, where the noncommutative multiplication is implemented.

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